Годишен преговор

Числови редици

19. $x_{n+1}=\dfrac{3^{n+1}}{(n+1)!}=\dfrac{3^n.3}{n!(n+1)}=\dfrac{3x_n}{n+1}$

$\Rightarrow\ x_{n+1}=\dfrac{3x_n}{n+1},\ \ x_1=3.$

Тъй като $a_n>0,\ \forall n$, то можем да разгледаме частното $\dfrac{x_{n+1}}{x_n}=\dfrac{3x_n}{(n+1)x_n}=\dfrac{3}{n+1}\geq 1$ при $n\geq 2$.

$\Rightarrow$ $\{x_n\}$ е монотонно намаляваща.

20. а) $\lim\limits_{n\to\infty}\dfrac{2n^4+3}{\sqrt{2}n^4-3}=\lim\limits_{n\to\infty}\dfrac{n^4\left(2+\dfrac{3}{n^4}\right)}{n^4\left(\sqrt{2}-\dfrac{3}{n^4}\right)}=\sqrt{2}$;

б) $\lim\limits_{n\to\infty}\dfrac{3n^3}{-(\sqrt{3}+1)n^4+4}=\lim\limits_{n\to\infty}\dfrac{3n^3}{n^4\left(-\sqrt{3}-1+\dfrac{4}{n^4}\right)}=0$;

в) $\lim\limits_{n\to\infty}\dfrac{2n^5}{-n^3+4}=\lim\limits_{n\to\infty}\dfrac{2n^5}{n^3\left(-1+\dfrac{4}{n^3}\right)}=-\infty$;

г) $\lim\limits_{n\to\infty}\dfrac{-2n-2}{\sqrt{n^2+1}+\sqrt{n^2}}=\lim\limits_{n\to\infty}\dfrac{n\left(-2-\dfrac{2}{n}\right)}{n\left(\sqrt{1+\dfrac{1}{n^2}}+1\right)}=-1$;

д) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n}+n^2}{\sqrt{n^2-3}}=\lim\limits_{n\to\infty}\dfrac{n^2\left(\dfrac{1}{n\sqrt{n}}+1\right)}{n\sqrt{1-\dfrac{3}{n^2}}}=\infty$;

е) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n+3}-\sqrt{n}}{2\sqrt{n}-\sqrt{n+1}}=\lim\limits_{n\to\infty}\dfrac{\sqrt{n}\left(\sqrt{1+\dfrac{3}{n}}-1\right)}{\sqrt{n}\left(2-\sqrt{1+\dfrac{1}{n}}\right)}=0$.

21. а) $\lim\limits_{a_n\to\sqrt{2}}\dfrac{a_n+3}{2a_n{}^2+4}=\dfrac{\sqrt{2}+3}{8}$;

б) $\lim\limits_{a_n\to-\sqrt{3}}\dfrac{a_n+1}{3a_n{}^2-1}=\dfrac{1-\sqrt{3}}{8}$.

в) $\lim\limits_{a_n\to 3}\dfrac{a_n{}^2-a_n-6}{3a_n{}^2-9a_n}=\lim\limits_{a_n\to 3}\dfrac{(a_n-3)(a_n+2)}{3a_n(a_n-3)}=\dfrac{5}{9}$.

22. Във всички подусловия на задачата $|q|<1$ $\Rightarrow$ прогресията е безкрайно намаляваща геометрична прогресия и $S=\dfrac{a_1}{1-q}$.

а) $S=\dfrac{a_1}{1-q}=\dfrac{1}{1-\dfrac{2}{\sqrt{5}}}=5+2\sqrt{5}$;

б) $S=\dfrac{a_1}{1-q}=\dfrac{24}{1-\dfrac{1}{3}}=36$;

в) $S=\dfrac{a_1}{1-q}=\dfrac{5}{4(1-0,4)}=\dfrac{25}{12}$;

г) $S=\dfrac{a_1}{1-q}=\dfrac{1}{1+\dfrac{1}{2}}=\dfrac{2}{3}$.

23. Сумата на всички членове след a_1 е равна на $S-a_1$.

$\Rightarrow$ $a_1=3(S-a_1)$.

Като използваме, че $S=\dfrac{a_1}{1-q}$, имаме $a_1=3\left(\dfrac{a_1}{1-q}-a_1\right)$.

Тъй като $a_1\neq 0$, то $1=3\left(\dfrac{1}{1-q}-1\right)$, откъдето $q=\dfrac{1}{4}$.

24. Съставяме системата

$\begin{cases} a_1+a_1q=8 \\ \dfrac{a_1}{1-q}=\dfrac{32}{3} \end{cases}$, която има две решения $a_1=\dfrac{16}{3}$, $q=\dfrac{1}{2}$ и $a_1=16$, $q=-\dfrac{1}{2}$.

Функции

25. а) $\lim\limits_{x\to 1}\dfrac{3x^2-x+5}{x+1}=\dfrac{7}{2}$;

б) $\lim\limits_{x\to -1}\dfrac{x^2+3x+2}{x+1}=\lim\limits_{x\to -1}\dfrac{(x+1)(x+2)}{x+1}=1$;

в) $\lim\limits_{x\to 2}\dfrac{\sqrt{x+3}-\sqrt{5}}{x^2-4}=\lim\limits_{x\to 2}\dfrac{x+3-5}{(x-2)(x+2)\left(\sqrt{x+3}+\sqrt{5}\right)}=\dfrac{\sqrt{5}}{40}$;

г) $\lim\limits_{x\to 2}\dfrac{2-\sqrt{x+2}}{\sqrt{x-1}-1}=\lim\limits_{x\to 2}\dfrac{-(x-2)\left(\sqrt{x-1}+1\right)}{(x-2)\left(2+\sqrt{x+2}\right)}=-\dfrac{1}{2}$;

д) $\lim\limits_{x\to 0}\dfrac{3-\sqrt{x+9}}{x}=\lim\limits_{x\to 0}\dfrac{x}{x\left(3+\sqrt{x+9}\right)}=-\dfrac{1}{6}$;

е) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-\sqrt{1-x}}{4x}=\lim\limits_{x\to 0}\dfrac{x+1-1+x}{4x\left(\sqrt{x+1}+\sqrt{1-x}\right)}=\dfrac{1}{4}$;

ж) $\lim\limits_{x\to 1}\left(\dfrac{2}{x^2-1}-\dfrac{3}{x^3-1}\right)=\lim\limits_{x\to 1}=\dfrac{2x^2-x-1}{(x-1)(x+1)(x^2+x+1)}=\dfrac{1}{2}$;

з) $\lim\limits_{x\to 0}\dfrac{x}{1-\sqrt{1-x}}=\lim\limits_{x\to 0}\dfrac{x\left(1+\sqrt{1-x}\right)}{x}=2$.

26. а) $\lim\limits_{x\to\infty}\dfrac{3x^3-2x}{2x^4-1}=\lim\limits_{x\to\infty}\dfrac{x^3\left(3-\dfrac{2}{x^2}\right)}{x^4\left(2-\dfrac{1}{x^4}\right)}=0$;

б) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x+2}+\sqrt{3x}}{\sqrt{5x}}=\lim\limits_{x\to\infty}\dfrac{\sqrt{x}\left(\sqrt{1+\dfrac{2}{x}}+\sqrt{3}\right)}{\sqrt{x}\sqrt{5}}=\dfrac{1+\sqrt{3}}{\sqrt{5}}$;

в) $\lim\limits_{x\to\infty}(\sqrt{x-1}-\sqrt{x})=\lim\limits_{x\to\infty}\dfrac{x-1-x}{\sqrt{x-1}+\sqrt{x}}=0$;

г) $\lim\limits_{x\to\infty}(x-\sqrt{x})=\dfrac{x^2-x}{x+\sqrt{x}}=\lim\limits_{x\to\infty}\dfrac{x^2\left(1-\dfrac{1}{x}\right)}{x\left(1+\dfrac{1}{\sqrt{x}}\right)}=\infty$;

д) $\lim\limits_{x\to\infty}\dfrac{3x^5-3x}{x^5+2}=\lim\limits_{x\to\infty}\dfrac{x^5\left(3-\dfrac{3}{x^2}\right)}{x^5\left(1+\dfrac{2}{x^5}\right)}=3$;

е) $\lim\limits_{x\to\infty}\dfrac{-2x^2+1}{3x}=\lim\limits_{x\to\infty}\dfrac{x^2\left(-2+\dfrac{1}{x^2}\right)}{3x}=-\infty$;

ж) $\lim\limits_{x\to\infty}\dfrac{-2x^3+1}{2x}=\lim\limits_{x\to\infty}\dfrac{x^3\left(-2+\dfrac{1}{x^3}\right)}{2x}=-\infty$;

з) $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}+\dfrac{2x}{1-x}\right)=1-1-2=-2$.

27. а) $\lim\limits_{x\to 0}\dfrac{\text{tg}x-\sin x}{2x^3}=\lim\limits_{x\to 0}\dfrac{\sin x(1-\cos x)}{2x^3\cos x}=\lim\limits_{x\to 0}\dfrac{\sin x.\sin^2\dfrac{x}{2}}{x^3\cos x}=\dfrac{1}{4}$;

б) $\lim\limits_{x\to 0}\dfrac{3x-\sin x}{x}=3-1=2$;

в) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{2x^2}:=\lim\limits_{x\to 0}\dfrac{\sin^2\dfrac{x}{2}}{x^2}=\dfrac{1}{4}$;

г) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sqrt{x+3}-\sqrt{3}}=\lim\limits_{x\to 0}\dfrac{\sin 2x.\left(\sqrt{x+3}+\sqrt{3}\right)}{x+3-3}=4\sqrt{3}$;

д) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\sin^2 x}=\lim\limits_{x\to 0}\dfrac{2\sin^2\dfrac{x}{2}}{\sin^2 x}=\dfrac{1}{2}$.

28. а) $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}=\lim\limits_{x\to a}\dfrac{2\cos\dfrac{x+a}{2}\sin\dfrac{x-a}{2}}{x-a}=\cos a$;

б) $\lim\limits_{x\to a}\dfrac{\cos x-\cos a}{2(x-a)}=\lim\limits_{x\to a}\dfrac{-\sin\dfrac{x+a}{2}\sin\dfrac{x-a}{2}}{(x-a)}=-\dfrac{\sin a}{2}$;

в) $\lim\limits_{x\to a}\dfrac{\text{tg}x-\text{tg}a}{x-a}=\lim\limits_{x\to a}\dfrac{\sin(x-a)}{(x-a)\cos x\cos a}=\dfrac{1}{\cos^2 a}$.

29. $\lim\limits_{x\to 0}\dfrac{ax^2+1}{b-x^2}=-\infty$ при $b=0$. При $b\neq 0$ границата е $\dfrac{1}{b}$.

$\lim\limits_{x\to\infty}\dfrac{ax^2+1}{b-x^2}=\lim\limits_{x\to\infty}\dfrac{ax^2+1}{-x^2}=\lim\limits_{x\to\infty}\dfrac{x^2\left(a+\dfrac{1}{x^2}\right)}{-x^2}=-a$. За да бъде тази граница равна на 3, то $a=-3$

Директно се проверява, че $a=-3$ и $b=0$ удовлетворяват условието на задачата.

30. В интервалите $x>0$ и $x<0$ функцията е непрекъсната.

$f(0)=-\dfrac{1}{3}$.

$\lim\limits_{\substack{x\to 0\\ x<0}}\dfrac{x}{2x-3}=0\neq-\dfrac{1}{3}=f(0)$ $\Rightarrow$ функцията е прекъсната при $x=0$.

31. $f(0)=a+1$

$\lim\limits_{\substack{x\to 0\\ x<0}}\dfrac{a\sin 2x}{3x}=\dfrac{2a}{3}$

$\lim\limits_{\substack{x\to 0\\ x>0}}\dfrac{a\sin 2x}{3x}=\dfrac{2a}{3}$

За да бъде непрекъсната функцията в точката $x=0$ трябва $\dfrac{2a}{3}=a+1$, $a=-3$.

Директно се проверява, че при $a=-3$ функцията е непрекъсната в точката $x=0$.

32. а) $f(x)=\frac{1}{x+1}-\frac{1}{x-1}+\frac{2}{x^2+1}=\frac{-2}{x^2-1}+\frac{2}{x^2+1}=\frac{-4}{x^4-1}$.

$f'(x)=\frac{4.4x^3}{(x^4-1)^2}=\frac{16x^3}{(x^4-1)^2}$.

33. а) $f'(x)=1-\frac{1}{8\sqrt{x}}-\frac{2}{5x^2}-2x^5$.

$f'(1)=1-\frac{1}{8}-\frac{2}{5}-2=-\frac{61}{40}$

б) $f'(x)=\frac{-24}{x^3}+\frac{4x^3}{4}+\frac{16}{x^5}+\frac{2}{3}.\frac{3}{2}.\sqrt{x}=-\frac{24}{x^3}+x^3+\frac{16}{x^5}+\sqrt{x}$.

$f'(1)=-24+1+16+1=-6$.

в) $f(x)=(x^2-3x+1)(2x+3)=2x^3-3x^2-7x+3$.

$f'(x)=6x^2-6x-7$.

$f'(-1)=6+6-7=5$.

г) $f'(x)=\sqrt{2x-3}+\frac{1}{2}.\frac{(x+2).2}{\sqrt{2x-3}}=\sqrt{2x-3}+\frac{x+2}{\sqrt{2x-3}}$.

$f'(2)=5$.

д) $f'(x)=\frac{-(1+x)-(1-x)}{(1+x)^2}=\frac{-2}{(1+x)^2}$.

$f'(-2)=-2$.

е) $f'(x)=\frac{-(2+x^2)-(1-x).2x}{(2+x^2)^2}=\frac{x^2-2x-2}{(2+x^2)^2}$.

$f'(0)=-\frac{1}{2}$.

ж) $f(x)=\frac{x^2-6x+9}{x}+6-\frac{4}{x}=x+\frac{5}{x}$.

$f'(x)=1-\frac{5}{x^2}$.

$f'(-5)=\frac{4}{5}$.

з) $f'(x)=2\cos 2x+\frac{2}{\sin^2 x}$

$f'(\frac{\pi}{2})=0$.

и) $f'(x)=\frac{\sin x(\cos x+1)+(1-\cos x)\sin x}{(\cos x+1)^2}=\frac{2\sin x}{(\cos x+1)^2}$.

$f'(\frac{\pi}{2})=2$.